

KODAIRA DIMENSION AND ZEROS OF HOLOMORPHIC ONE-FORMS

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ABSTRACT. We show that every holomorphic one-form on a smooth complex projective variety of general type must vanish at some point. The proof uses generic vanishing theory for Hodge \mathscr{D} -modules on abelian varieties.

INTRODUCTION

1. In this paper, we use results about Hodge \mathscr{D} -modules on abelian varieties [PS11] to prove two conjectures about zeros of holomorphic one-forms on smooth complex projective varieties. One conjecture was formulated by Hacon and Kovács [HK05] and by Luo and Zhang [LZ05]; the former partly attribute the question to Carrell [Car74], and also explain why it is natural to consider varieties of general type.

Conjecture 1.1 (Hacon-Kovács, Luo-Zhang). *If X is a smooth complex projective variety of general type, then the zero locus*

$$Z(\omega) = \{ x \in X \mid \omega(T_x X) = 0 \}$$

of every global holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ is nonempty.

The statement is a tautology in the case of curves; for surfaces, it was proved by Carrell [Car74], and for threefolds by Luo and Zhang [LZ05]. It was also known to be true if the canonical bundle of X is ample [Zha97], and more generally when X is minimal [HK05]. For varieties that are not necessarily of general type, Luo and Zhang proposed the following more general version of the conjecture, in terms of the Kodaira dimension $\kappa(X)$, and proved it in the case of threefolds [LZ05].

Conjecture 1.2 (Luo-Zhang). *Let X be a smooth complex projective variety, and let $W \subseteq H^0(X, \Omega_X^1)$ be a linear subspace such that $Z(\omega)$ is empty for every nonzero one-form $\omega \in W$. Then the dimension of W can be at most $\dim X - \kappa(X)$.*

2. Since every holomorphic one-form on X is the pullback of a holomorphic one-form from the Albanese variety $\text{Alb } X$, it is natural to consider an arbitrary morphism from X to an abelian variety, and to ask under what conditions the pullback of a holomorphic one-form must have a zero at some point of X . Our main result is the following theorem, which implies both of the above conjectures.

Theorem 2.1. *Let X be a smooth complex projective variety, and let $f: X \rightarrow A$ be a morphism to an abelian variety. If the Iitaka model of X dominates $f(X)$, then $Z(\omega)$ is nonempty for every ω in the image of $f^*: H^0(A, \Omega_A^1) \rightarrow H^0(X, \Omega_X^1)$.*

The condition on the Iitaka model means that, up to birational equivalence, the morphism f should factor through the Iitaka fibration of (X, K_X) .

3. One consequence of [Conjecture 1.2](#) is the following.

Corollary 3.1. *If $f: X \rightarrow A$ is a smooth morphism from a smooth complex projective variety onto an abelian variety, then $\dim A \leq \dim X - \kappa(X)$.*

In particular, there are no nontrivial smooth morphisms from a variety of general type to an abelian variety; this was proved by Viehweg and Zuo [\[VZ01\]](#) when the base is an elliptic curve, and then by Hacon and Kovács [\[HK05\]](#) in general. Together with subadditivity, [Corollary 3.1](#) has the following application.

Corollary 3.2. *If $f: X \rightarrow A$ is a smooth morphism onto an abelian variety, and if all fibers of f are of general type, then f is birationally isotrivial.*

Proof. We have to show that X becomes birational to a product after a generically finite base-change; it suffices to prove that $\text{Var}(f) = 0$, in Viehweg’s terminology. Since the fibers of f are of general type, we know from the main result of [\[Kol\]](#) that

$$\kappa(F) + \text{Var}(f) \leq \kappa(X).$$

On the other hand, $\kappa(X) \leq \dim F$ by [Corollary 3.1](#), and so we are done. \square

Successive versions of this result go back to Migliorini [\[Mig95\]](#) (for smooth families of minimal surfaces of general type over an elliptic curve), Kovács [\[Kov96\]](#) (for smooth families of minimal varieties of general type over an elliptic curve), and Viehweg-Zuo [\[VZ01\]](#) (for smooth families of varieties of general type over an elliptic curve). Over a base of arbitrary dimension, Kovács [\[Kov97\]](#) and Zhang [\[Zha97\]](#) have shown that if the canonical bundles of the fibers is assumed to be ample, the morphism f must actually be isotrivial.

4. Before we get into the details of the proof, we explain how to deduce [Conjecture 1.2](#) – and hence also [Conjecture 1.1](#), which is a special case – from [Theorem 2.1](#).

The statement of the conjecture is vacuous when $\kappa(X) = -\infty$, and so we shall assume from now on that $\kappa(X) \geq 0$. Let $\mu: X' \rightarrow X$ be a birational modification of X , such that $g: X' \rightarrow Z$ is a smooth model for the Iitaka fibration. The general fiber of g is then a smooth projective variety of dimension $\delta(X) = \dim X - \kappa(X)$ and Kodaira dimension zero, and therefore maps surjectively to its own Albanese variety [\[Kaw81, Theorem 1\]](#). By a standard argument [\[Kaw81, Proof of Theorem 13\]](#), the image in $\text{Alb } X$ of every fiber of g must be a translate of a single abelian variety. Letting A denote the quotient of $\text{Alb } X$ by this abelian variety, we obtain $\dim A \geq \dim H^0(X, \Omega_X^1) - \delta(X)$, and the following commutative diagram:

$$(4.1) \quad \begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ g \downarrow & & \downarrow f \\ Z & \longrightarrow & A \end{array}$$

By construction, Z dominates $f(X)$, and so we conclude from [Theorem 2.1](#) that all holomorphic one-forms in the image of $f^*: H^0(A, \Omega_A^1) \rightarrow H^0(X, \Omega_X^1)$ have a nonempty zero locus. Since this subspace has codimension at most $\delta(X)$ in the space of all holomorphic one-forms, we see that [Conjecture 1.2](#) must be true.

PROOF OF THE THEOREM

5. Let $f: X \rightarrow A$ be a morphism from a smooth projective variety to an abelian variety. We define $V = H^0(A, \Omega_A^1)$, and introduce the set

$$Z_f = \{ (x, \omega) \in X \times V \mid (f^*\omega)(T_x X) = 0 \}.$$

To prove [Theorem 2.1](#), we have to show that the projection from Z_f to V is surjective; what we will actually prove is that $(f \times \text{id})(Z_f) \subseteq A \times V$ maps onto V .

6. Our method is inspired by the paper [\[VZ01\]](#), in which Viehweg and Zuo show, among other things, that there are no smooth morphisms from a complex projective variety X of general type to an elliptic curve E . To orient the reader, we shall briefly recall a key step in their proof [\[VZ01, Lemma 3.1\]](#); a similar technique also appears in [\[Kov96, Kov02\]](#).

Using a carefully-chosen resolution of singularities of a certain branched covering of X , Viehweg and Zuo produce a logarithmic Higgs bundle $\bigoplus \mathcal{E}^{p,q}$ on the elliptic curve E , whose Higgs field

$$\theta^{p,q}: \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p-1,q+1} \otimes \Omega_E^1(\log S')$$

has logarithmic poles along a divisor S' ; by construction, S' contains the set of points S where the morphism from X to E is singular. They also construct a Higgs subbundle $\bigoplus \mathcal{F}^{p,q}$ with two properties: the restriction of the Higgs field satisfies

$$\theta^{p,q}(\mathcal{F}^{p,q}) \subseteq \mathcal{F}^{p-1,q+1} \otimes \Omega_E^1(\log S),$$

and $\mathcal{F}^{r,0}$ is an ample line bundle (where $r = \dim X - 1$). Since any subbundle of $\ker \theta^{p,q}$ has degree ≤ 0 , it follows that S cannot be empty: otherwise, the image of $\mathcal{F}^{r,0}$ under some iterate of the Higgs field would be an ample subbundle of $\ker \theta^{p,q}$.

7. A natural higher-dimensional generalization of a Higgs bundle is the associated graded of a Hodge \mathcal{D} -module [\[Sai88\]](#). Recall that a Hodge \mathcal{D} -module on a smooth complex algebraic variety X is a very special kind of filtered \mathcal{D} -module (\mathcal{M}, F) ; in particular, \mathcal{M} is a regular holonomic left \mathcal{D}_X -module, and $F_\bullet \mathcal{M}$ is a good filtration by \mathcal{O}_X -coherent subsheaves. The associated graded

$$\text{gr}_\bullet^F \mathcal{M} = \bigoplus_{k \in \mathbb{Z}} F_k \mathcal{M} / F_{k-1} \mathcal{M}$$

is coherent over the symmetric algebra $\text{Sym } \mathcal{T}_X$. It therefore determines a coherent sheaf $\text{gr}^F \mathcal{M}$ on the cotangent bundle T^*X , whose support is the characteristic variety $\text{Ch}(\mathcal{M})$ of the \mathcal{D} -module. A longer summary of Saito's theory can be found in [\[PS11, Sections 5 and 6\]](#).

8. For later use, we point out another connection between [Theorem 2.1](#) and Hodge \mathcal{D} -modules. It has to do with the structure of the set $(f \times \text{id})(Z_f)$.

The structure sheaf \mathcal{O}_X is naturally a left \mathcal{D}_X -module; the direct image functor for \mathcal{D} -modules takes it to a complex $f_+ \mathcal{O}_X$ of regular holonomic \mathcal{D}_A -modules. According to Kashiwara's estimate for the behavior of the characteristic variety,

$$\text{Ch}(f_+ \mathcal{O}_X) \subseteq (f \times \text{id})(df^{-1}(0)) = (f \times \text{id})(Z_f),$$

where the notation is as in the following diagram:

$$(8.1) \quad \begin{array}{ccc} X \times V & \xrightarrow{df} & T^*X \\ \downarrow f \times \text{id} & & \\ A \times V & & \end{array}$$

One consequence of Saito's theory is that \mathcal{O}_X , equipped with the obvious filtration ($F_0\mathcal{O}_X = \mathcal{O}_X$), is actually a Hodge \mathcal{D} -module. Because f is projective, $f_+\mathcal{O}_X$ (with the induced filtration) is then also a complex of Hodge \mathcal{D} -modules on A , and Saito's version of the Decomposition Theorem gives us a non-canonical splitting

$$(8.2) \quad f_+\mathcal{O}_X \simeq \bigoplus_{i \in \mathbb{Z}} (\mathcal{H}^i f_+\mathcal{O}_X)[-i]$$

in the derived category of Hodge \mathcal{D} -modules. This means that the set $(f \times \text{id})(Z_f)$ contains the characteristic varieties of the Hodge \mathcal{D} -modules $\mathcal{H}^i f_+\mathcal{O}_X$, for $i \in \mathbb{Z}$.

9. The above considerations suggest a way to generalize the construction in [VZ01] to the setting of [Theorem 2.1](#). Namely, suppose that we manage to find a Hodge \mathcal{D} -module (\mathcal{M}, F) on the abelian variety A , and a graded $\text{Sym } \mathcal{T}_A$ -submodule

$$\mathcal{F}_\bullet \subseteq \text{gr}_\bullet^F \mathcal{M}.$$

Denote by \mathcal{F} and $\text{gr}^F \mathcal{M}$ the associated coherent sheaves on $T^*A = A \times V$, and suppose that the following three conditions are satisfied:

- (1) There is a morphism $h: Y \rightarrow A$ from a smooth projective variety, such that (\mathcal{M}, F) is a direct summand of some $\mathcal{H}^i h_+\mathcal{O}_Y$ (with the induced filtration).
- (2) The support of \mathcal{F} is contained in the set $(f \times \text{id})(Z_f) \subseteq A \times V$.
- (3) For some $k \in \mathbb{Z}$, the sheaf \mathcal{F}_k is isomorphic to $L \otimes f_*\mathcal{O}_X$, where L is an ample line bundle on A .

If that is the case, we can use the generic vanishing theory for Hodge \mathcal{D} -modules that we developed in [PS11] to show that Z_f projects onto V .

Proposition 9.1. *If a Hodge \mathcal{D} -module (\mathcal{M}, F) and a graded $\text{Sym } \mathcal{T}_A$ -module \mathcal{F}_\bullet with the above properties exist, then the projection from Z_f to V must be surjective.*

Proof. Let P be the normalized Poincaré bundle on $A \times \hat{A}$. Using its pull-back to $A \times \hat{A} \times V$ as a kernel, we define the Fourier-Mukai transform of $\text{gr}^F \mathcal{M}$ to be the complex of coherent sheaves

$$E = \mathbf{R}\Phi_P(\text{gr}^F \mathcal{M}) = \mathbf{R}(p_{23})_* \left(p_{13}^*(\text{gr}^F \mathcal{M}) \otimes p_{12}^* P \right)$$

on $\hat{A} \times V$. Because (\mathcal{M}, F) is a Hodge \mathcal{D} -module of geometric origin by (1), both E and the dual complex $\mathbf{R}\mathcal{H}om(E, \mathcal{O})$ have the following two properties:

- (a) They are *perverse coherent sheaves*, meaning that the support of the ℓ -th cohomology sheaf has codimension at least 2ℓ in $\hat{A} \times V$ [PS11, Theorem 15.2].
- (b) The union of the supports of all the higher cohomology sheaves is a finite union of translates of triple tori [PS11, Proof of Proposition 13.2].

Here a triple torus, in Simpson's terminology, means a subset of the form

$$\text{im} \left(\varphi^*: \hat{B} \times H^0(B, \Omega_B^1) \rightarrow \hat{A} \times H^0(A, \Omega_A^1) \right),$$

where $\varphi: A \rightarrow B$ is a morphism to another abelian variety. It follows that the 0-th cohomology sheaf of the complex E is locally free outside a finite union of translates of triple tori of codimension at least two; in particular, the restriction of E to the subspace $\{\alpha\} \times V$ is a locally free sheaf for general $\alpha \in \hat{A}$. Translated back to a statement on $A \times V$, this means that the pushforward

$$q_*(p^*P_\alpha \otimes \mathrm{gr}^F \mathcal{M})$$

is a locally free sheaf on V ; here $p: A \times V \rightarrow A$ and $q: A \times V \rightarrow V$ denote the projections to the two factors.

Now suppose that the projection from Z_f to V was *not* surjective. Because the support of \mathcal{F} is contained in $(f \times \mathrm{id})(Z_f)$ by (2), the subsheaf

$$q_*(p^*P_\alpha \otimes \mathcal{F}) \subseteq q_*(p^*P_\alpha \otimes \mathrm{gr}^F \mathcal{M})$$

would then be torsion, and therefore zero. In other words, we would have

$$H^0(A, P_\alpha \otimes \mathcal{F}_k) = 0$$

for general $\alpha \in \hat{A}$, and every $k \in \mathbb{Z}$; but this is not possible because of (3). \square

10. Note that (b) relies on the Decomposition Theorem (8.2). The Fourier-Mukai transform of the complex $\mathrm{gr}^F(h_+ \mathcal{O}_Y)$, which equals that of \mathcal{O}_Y , has this property by a result of Arapura and Simpson about cohomology support loci for Higgs bundles [Ara92]. But in order to say something about direct summands of individual cohomology sheaves, one needs to know that $h_+ \mathcal{O}_Y$ splits in the derived category.

11. For the remainder of the paper, we let X be a smooth complex projective variety of dimension n and Kodaira dimension $\kappa(X) \geq 0$. We also assume that we have a morphism $f: X \rightarrow A$ to an abelian variety, such that the Iitaka model of X dominates $f(X)$. This means that we can find a commutative diagram as in (4.1), where $g: X' \rightarrow Z$ is a smooth model for the Iitaka fibration of X , and $\mu: X' \rightarrow X$ is a birational morphism. A simple consequence is that if L is an ample line bundle on A , then $\omega_X^d \otimes f^*L^{-1}$ has a section for some $d \gg 0$. Using the geometry of abelian varieties, this can be improved as follows.

Lemma 11.1. *After a finite étale base change on A , we can find an ample line bundle L such that the d -th power of $\omega_X \otimes f^*L^{-1}$ has a section for some $d \geq 1$.*

Proof. Fix an ample line bundle L_1 on A , and choose $d \geq 1$ such that $\omega_X^d \otimes f^*L_1^{-1}$ has a section. Let $[d]: A \rightarrow A$ denote multiplication by d ; then $[d]^*L_1 = L^d$ for some ample line bundle L on A , which clearly does the job. \square

Since the conclusion of Theorem 2.1 is unaffected by finite étale morphisms, we may assume for the remainder of the argument that the d -th power of the line bundle $B = \omega_X \otimes f^*L^{-1}$ has a nontrivial section; for the sake of convenience, we shall take $d \geq 1$ to be the smallest integer with this property.

12. Any nontrivial section $s \in H^0(X, B^d)$ defines a branched covering $\pi: X_d \rightarrow X$ of degree d , unramified outside the divisor $Z(s)$; see [EV92, §3] for details. Since d is minimal, X_d is irreducible; let $\mu: Y \rightarrow X_d$ be a resolution of singularities that is

an isomorphism over the complement of $Z(s)$, and define $\varphi = \pi \circ \mu$ and $h = f \circ \varphi$. The following commutative diagram shows all the relevant morphisms:

$$\begin{array}{ccccc}
 & & \varphi & & \\
 & \nearrow & & \searrow & \\
 Y & \xrightarrow{\mu} & X_d & \xrightarrow{\pi} & X \\
 & \searrow & & \downarrow f & \\
 & & & A &
 \end{array}$$

(The diagram shows a commutative diagram with nodes Y, X_d, X, and A. Morphisms are: Y → X_d (μ), X_d → X (π), X → A (f), Y → A (h), and a curved arrow Y → X (φ) above the top row.)

By construction, X_d is embedded in the total space of the line bundle B , and so the pullback π^*B has a tautological section; the induced morphism $\varphi^*B^{-1} \rightarrow \mathcal{O}_Y$ is an isomorphism over the complement of $Z(s)$. After composing it with $\varphi^*\Omega_X^k \rightarrow \Omega_Y^k$, we obtain for every $k = 0, 1, \dots, n$ an injective morphism

$$\varphi^*(B^{-1} \otimes \Omega_X^k) \rightarrow \Omega_Y^k,$$

which is actually an isomorphism over the complement of $Z(s)$. Pushing forward to X , and using the fact that $\mathcal{O}_X \rightarrow \varphi_*\mathcal{O}_Y$ is injective, we find that the morphisms

$$(12.1) \quad B^{-1} \otimes \Omega_X^k \rightarrow \varphi_*\Omega_Y^k$$

are also injective.

13. Let $S = \text{Sym } V^*$ be the symmetric algebra on the dual of $V = H^0(A, \Omega_A^1)$, and consider the complex of graded $\mathcal{O}_X \otimes S$ -modules

$$C_{X,\bullet} = \left[\mathcal{O}_X \otimes S_{\bullet-g} \rightarrow \Omega_X^1 \otimes S_{\bullet-g+1} \rightarrow \cdots \rightarrow \Omega_X^n \otimes S_{\bullet-g+n} \right],$$

placed in cohomological degrees $-g, \dots, 0$, where $g = \dim A$ and $n = \dim X$. The differential in the complex is induced by the evaluation morphism $V \otimes \mathcal{O}_X \rightarrow \Omega_X^1$. Concretely, let $\omega_1, \dots, \omega_g \in V$ be a basis, and denote by $s_1, \dots, s_g \in S_1$ the dual basis; then the formula for the differential is

$$\Omega_X^p \otimes S_{\bullet-g+p} \rightarrow \Omega_X^{p+1} \otimes S_{\bullet-g+p+1}, \quad \theta \otimes s \mapsto \sum_{i=1}^g (\theta \wedge f^*\omega_i) \otimes s_i s.$$

We use similar notation on Y as well.

Lemma 13.1. *There is a morphism of complexes of graded $\mathcal{O}_A \otimes S$ -modules*

$$\mathbf{R}f_*(B^{-1} \otimes C_{X,\bullet}) \rightarrow \mathbf{R}h_*C_{Y,\bullet},$$

induced by the individual morphisms in (12.1).

Proof. The morphisms in (12.1) commute with the differentials in the two complexes because $\varphi^*(f^*\omega) = h^*\omega$ for every $\omega \in V$. \square

14. We denote by C_X the complex of coherent sheaves on $X \times V$ associated with the complex of graded $\mathcal{O}_X \otimes S$ -modules $C_{X,\bullet}$.

Lemma 14.1. *The support of C_X is equal to $Z_f \subseteq X \times V$.*

Proof. Let $p: X \times V \rightarrow X$ denote the first projection; then

$$C_X = \left[p^*\mathcal{O}_X \rightarrow p^*\Omega_X^1 \rightarrow \cdots \rightarrow p^*\Omega_X^n \right],$$

with differential induced by the tautological section of $p^*\Omega_X^1$. This shows that C_X is equal to the pullback of the Koszul resolution for the structure sheaf of the zero

section in T^*X via the morphism $df: X \times V \rightarrow T^*X$. In particular, $\text{Supp } C_X$ is equal to $df^{-1}(0) = Z_f$, in the notation of (8.1). \square

15. We are now in a position to carry out the construction announced in §9.

For the Hodge \mathcal{D} -module (\mathcal{M}, F) , we take $\mathcal{H}^0 h_+ \mathcal{O}_Y$ with the induced filtration; it is again a Hodge \mathcal{D} -module because the morphism h is projective. As explained in [PS11, Section 6], one has a canonical isomorphism

$$\text{gr}_{\bullet}^F(h_+ \mathcal{O}_Y) = \mathbf{R}h_* C_{Y, \bullet}$$

as complexes of graded modules over $\text{Sym } \mathcal{T}_A = \mathcal{O}_A \otimes S$. Roughly speaking, this means that taking the associated graded is compatible with projective direct images – one of the most useful general properties of Hodge \mathcal{D} -modules. Returning to the construction, the associated graded of the Hodge \mathcal{D} -module $\mathcal{H}^i h_+ \mathcal{O}_Y$ is isomorphic to the i -th cohomology sheaf of the complex $\mathbf{R}h_* C_{Y, \bullet}$, and in particular,

$$\text{gr}_{\bullet}^F \mathcal{M} = R^0 h_* C_{Y, \bullet}$$

as graded $\text{Sym } \mathcal{T}_A$ -modules. Now define \mathcal{F}_{\bullet} to be the image of $R^0 f_*(B^{-1} \otimes C_{X, \bullet})$ in $R^0 h_* C_{Y, \bullet}$; in this way, get a graded $\text{Sym } \mathcal{T}_A$ -submodule of $\text{gr}_{\bullet}^F \mathcal{M}$.

16. It remains to check that (\mathcal{M}, F) and \mathcal{F}_{\bullet} have all the required properties.

Proposition 16.1. *(\mathcal{M}, F) and \mathcal{F}_{\bullet} satisfy the conditions in (1)–(3).*

Proof. It is obvious from the construction that (1) holds. Because of Lemma 14.1, the support of \mathcal{F} is contained in $(f \times \text{id})(Z_f)$, which proves (2). It remains to show that (3) is true for $k = g - n$. We clearly have $C_{X, k} = \omega_X$ and $C_{Y, k} = \omega_Y$, and the morphism $f^* L = B^{-1} \otimes \omega_X \rightarrow \varphi_* \omega_Y$ in (12.1) is injective. After pushing forward to A , we find that the resulting morphism

$$L \otimes f_* \mathcal{O}_X = f_* f^* L \rightarrow h_* \omega_Y$$

is still injective. But $h_* \omega_Y = \text{gr}_k^F \mathcal{M}$, and so \mathcal{F}_k is isomorphic to $L \otimes f_* \mathcal{O}_X$. \square

Now Proposition 9.1 shows that Z_f projects onto V ; this means that every holomorphic one-form in the image of $f^*: H^0(A, \Omega_A^1) \rightarrow H^0(X, \Omega_X^1)$ has a nonempty zero locus. We have proved Theorem 2.1, Conjecture 1.2, and Conjecture 1.1.

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